# The philosophy behind loop quantum gravity

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Quantum gravity is expected to reconcile general relativity and quantum mechanics.

Because we have no experimental data, approaches are motivated by intellectual prejudices. What should we take as guiding principles?

Loop quantum gravity is a proposal for a non-perturbative and background independent quantization of general relativity.

Where does it stand with respect to the “problem” of time? the issue of background independence? the resolution of singularities? the nature of space-time?

A brief history of quantum gravity:

- 1952 Flat space quantization (Rosenfeld, Pauli, Fierz, Gupta,. . . )
- 1959 Canonical structure of general relativity (Dirac, Bergmann, Arnowit, Deser, Misner)
- 1964 Penrose introduces the idea of spin networks
- 1967 Wheeler-DeWitt equation
- 1974 Hawking radiation and black hole entropy
- 1984 String theory
- 1986 New variables for general relativity (Ashtekar, Sen)
- 1988 Loop representation and solutions to the Wheeler-DeWitt equation (Jacobson, Smolin)
- 1989 Extra dimensions from string theory
- 1995 Hilbert space of loop quantum gravity, geometric operators
- 2000' Spin foam models, group field theory, loop quantum cosmology,. . .

What are the relevant degrees of freedom around the Planck scale? geometry? topology? dimensionality?

What is the meaning of background independence? We have to choose

- space-time $\mathcal{M}$ = dimension, differential structure, topology, signature,
- metric $g_{\mu\nu}$ and matter fields $T_{\mu\nu}$.

A physical space-time is an equivalence class $[\mathcal{M}, g_{\mu\nu}, T_{\mu\nu}] = \dfrac{(\mathcal{M}, g_{\mu\nu}, T_{\mu\nu})}{\mathrm{Diff}(\mathcal{M})}$.

Appart from the information in $\mathcal{M}$, general relativity is a relational theory. The rest of the information is encoded in the **causal structure** and the **4-volume**.

For a given space-time $\mathcal{M}$, loop quantum gravity is able to provide a quantum mechanical description of the gravitational field. It is a relational theory in the same sense as classical general relativity.

The Hamiltonian of any generally covariant theory is vanishing as a sum of constraints. As John Stachel would put it, “There is no kinematics without dynamics”.

To compute observables of the gravitational field itself, we cannot use background dependent methods and fix the gravitational degrees of freedom.

How to build a background-independent quantum field theory of the gravitational field?

If geometry and matter are on the same footing we have to face some questions. Can we find atoms of geometry? Does geometry have constituents? What are its elementary excitations? Is the continuum picture around us only an approximation?

Historically, loop quantum gravity was developped as a canonical quantization of general relativity. Let us review the fundamental steps of this construction.

A first remark:

$$\begin{array}{ccc} \text{canonical formulation 1} & \simeq & \text{canonical formulation 2} \\ \downarrow & & \downarrow \\ \text{quantum theory 1} & \neq & \text{quantum theory 2} \end{array}$$

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The Einstein-Hilbert action

$$S_{\mathrm{EH}}[g] = \int_{\mathcal{M}} d^4x \sqrt{-g} R$$

written in terms of a local flat frame (tetrad)

$$e^I = e^I_\mu dx^\mu, \qquad g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{\mu\nu},$$

and an $\mathfrak{so}(3,1)$ spin connection $\omega^{IJ}$ with curvature $F_{IJ}$, becomes the so-called Hilbert-Palatini action

$$S_{\mathrm{HP}}[e,\omega] = \int_{\mathcal{M}} \star(e^I \wedge e^J) \wedge F_{IJ}[\omega].$$

The starting point for LQG is in fact the Holst action

$$S_\gamma[e,\omega] = \int_{\mathcal{M}} \left[ \star(e^I \wedge e^J) + \frac{1}{\gamma}(e^I \wedge e^J) \right] \wedge F_{IJ}[\omega],$$

which depends on a free parameter $\gamma \in \mathbb{R}^*$.

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Think about electromagnetism: $S[A] = -\frac{1}{4}\int_{\mathbb{M}} d^4x\, F^{\mu\nu} F_{\mu\nu}$.

The gauge group of this theory is $U(1)$. If we move a particle around a loop $\ell$, its wavefunction picks up a phase

$$\exp\left(-i \oint_{\ell} A\right) \in U(1).$$

In the Hamiltonian framework ($\mathcal{M} = \Sigma \times \mathbb{R}$), we introduce the Ashtekar variables

$$A_a^i = \omega_a^i + \gamma \omega_a^{0i}, \qquad E_i^a = \frac{1}{2}\epsilon^{abc}\epsilon_{ijk} e_b^j e_c^k,$$

$$\{A_a^i(x), E_j^b(y)\} = \gamma \delta_j^i \delta_a^b \delta^3(x,y).$$

The constraints are

$$G_i = D_a E_i^a, \qquad H_a = E_i^b F_{ab}^i, \qquad H = \frac{E_i^a E_j^b}{\sqrt{\det E}}\left(\epsilon^{ij}{}_k F_{ab}^k + 2(\gamma^2+1) K_{[a}^i K_{b]}^j\right).$$

They generate SU(2) gauge and spatial diffeomorphism symmetries, and time reparametrization (dynamics).

Phase space variables are the $\mathfrak{su}(2)$ connection $A_a^i$ and the electric field $E_i^a$.

The quantization strategy (Dirac) is

$$\mathcal{H}_{\text{kin}} \xrightarrow{\widehat{G}_i} \mathcal{H}_0 \xrightarrow{\widehat{H}_a} \mathcal{H} \xrightarrow{\widehat{H}} \mathcal{H}_{\text{phys}}.$$

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Canonical quantization:

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- Represent $\mathfrak{a}$ by operators living in a suitable Hilbert space.
- Turn the constraints into self-adjoint operators.
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- Choose a state $F$ on $\mathfrak{a}$, i.e. a positive linear functional

$$\begin{aligned} F : \mathfrak{a} &\rightarrow \mathbb{C} \\ a &\mapsto F(a) \end{aligned}$$

such that

$$F(a + \lambda b) = F(a) + \lambda F, \qquad F(\mathbb{I}) = 1, \qquad F(a^\star a) \geq 0.$$

- In usual quantum field theory on flat space-time, we use Poincaré invariance to find $F$ and construct the Fock space with Gaussian measure. This requires the use of a background metric.

The case for quantum geometrodynamics. 0904.0184 [gr-qc]

The algebra $\mathfrak{a}$ is generated by (without background metric)

$$\hat{q}(f) = \int_\Sigma d^3x\, f^{ab} q_{ab}, \qquad \hat{p}(g) = \int_\Sigma d^3x\, g_{ab} p^{ab},$$

$$[\hat{q}(f), \hat{p}(g)] = i\hbar \int_\Sigma d^3x\, f^{ab} g_{ab}.$$

A diffeomorphism $\phi$ acting on $f$ or $g$ induces an automorphism $\theta_\phi$ on $\mathfrak{a}$.

To fully implement background independence, we need a diffeomorphism invariant state $F$, i.e. such that $F\big(\theta_\phi(a)\big) = F(a)$. We would then have a diffeomorphism invariant representation of $\mathfrak{a}$ with a unitary action of the $\phi$ thanks to the GNS construction.

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The case for loop quantum gravity.

Remember that we have $\{A_a^i(x), E_j^b(y)\} = \gamma \delta_j^i \delta_a^b \delta^3(x,y)$.

If we construct the algebra $\mathfrak{a}$ with

$$\hat{A}(f) = \int_\Sigma d^3x\, A_a^i f_i^a, \qquad \hat{E}(g) = \int_\Sigma d^3x\, E_i^a g_a^i,$$

again, there is no diffeomorphism invariant state! Also, we cannot properly implement the SU(2) gauge invariance.

Appropriate smearing of the basic variables:

- The 2-form $E^a_i$ is smeared along a surface to define the electric flux

$$E_S = \int_S d^2\sigma \, E^a_i g^i_a.$$

- The connection 1-form $A^i_a$ is smeared along a path $\ell$ to define an SU(2) group element, the holonomy (this is a choice!)

$$h_\ell[A] = \mathcal{P} \exp \int_\ell A^i_a \tau_i \dot{\ell}^a.$$

This leads to the holonomy-flux algebra $\mathfrak{hf}$. Captures a finite number of degrees of freedom of the gravitational field.

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LOST (Lewandowski, Okolow, Sahlmann, Thiemann) theorem: On the holonomy-flux algebra, there is a unique SU(2)-gauge and diffeomorphism invariant state $F$.

The GNS construction then provides a Hilbert space $\mathcal{H}$ and a representation of the holonomy-flux algebra by quantum operators on it.

Uniqueness of the kinematical structure of loop quantum gravity.

A key role is played by the requirement of diffeomorphism invariance.

(More details in the section on loop quantum cosmology. . . )

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(More details in the section on loop quantum cosmology...)

A spin network is

- a graph $\Gamma$ consisting of $L$ links and $N$ nodes,
- a labelling of the links $\ell$ with SU(2) irreducible representations $j_\ell$,
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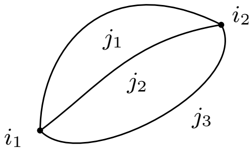


SU(2)-gauge and diffeomorphism invariant kinematical states are given by spin network states

$$|S\rangle = \bigotimes_\ell D^{(j_\ell)}(h_\ell[A]) \cdot \bigotimes_n i_n.$$

They form a basis of the loop quantum gravity Hilbert space

$$\mathcal{H} = \bigoplus_\Gamma \mathcal{H}_\Gamma = \bigoplus_\Gamma L^2\left[\mathrm{SU}(2)^L/\mathrm{SU}(2)^N\right].$$

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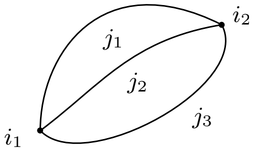


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Spin network states provide a notion of quantum Riemannian 3-geometry.

The simplest operator (partial observable in fact) that acts on a spin network state is the area operator

$$\widehat{\mathrm{Ar}}|\mathcal{S}\rangle \propto \gamma l_{\mathrm{Pl}}^2 \sqrt{j_\ell(j_\ell+1)}|\mathcal{S}\rangle.$$

The area of a surface punctured by the link $\ell$ has a discrete spectrum!

Geometrical operators have discrete spectra in loop quantum gravity: space is fundamentally discrete at the Planck scale.

UV finite in the sense that there are no transplankian degrees of freedom.

If the continuum around us is only a coarse-grained approximation, how does it arise physically? Shall we start from a quantum field theory for the building blocks of quantum gravity?

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Problem: define the action of the Hamiltonian constraint, $\widehat{H}\Psi = 0$.

Also, how to compute the dynamic and extract physics? Already a problem in classical general relativity.

Recall that a Dirac observable $\mathcal{O}$ commutes with the canonical Hamiltonian:

$$\{\mathcal{O}, H + H_a\} = 0.$$

There must be a physical Hamiltonian $H_{\text{phys}} \neq 0$, generating the dynamics of $\mathcal{O}$. For this, we can add matter and define reference frames.

Reduced phase space approach:

- Construct Dirac observables.
- Derive the "true" Hamiltonian $H_{\text{phys}}$ by deparametrizing the system.
- Compute $\dot{\mathcal{O}} = \{\mathcal{O}, H_{\text{phys}}\}$ and then quantize.

Fully relational dynamics.

Remark: We can also derive the black hole entropy formula without the dynamics.

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What about the covariant theory?

Spin foam models allow to define the gravitational boundary path integral (statement of intent!)

$$\langle \Sigma_1, q_1 | \mathcal{P}_H | \Sigma_2, q_2 \rangle_{\text{kin}} = \langle \Sigma_1, q_1 | \Sigma_2, q_2 \rangle_{\text{phys}} = \int_{g_{|\Sigma} = q} \mathcal{D}[g] \exp(i S_{\text{EH}}),$$

where $S_{\text{EH}}$ is the Einstein-Hilbert action and $|\Sigma, q\rangle$ is a kinematical state of spatial 3-geometry.

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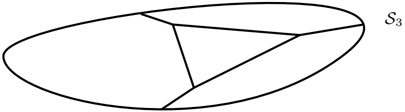
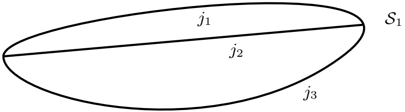
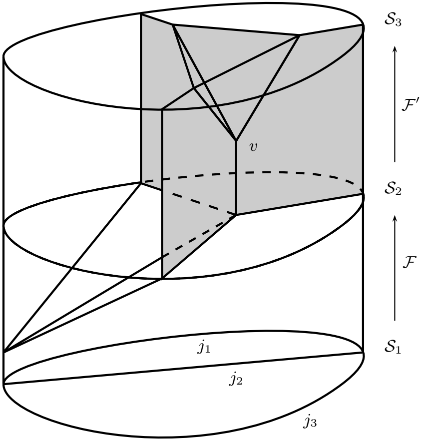
Let us start with two boundary states of the gravitational field.



A spin foam $\mathcal{F} : \mathcal{S}_1 \to \mathcal{S}_2$ is a sum over spin network histories.



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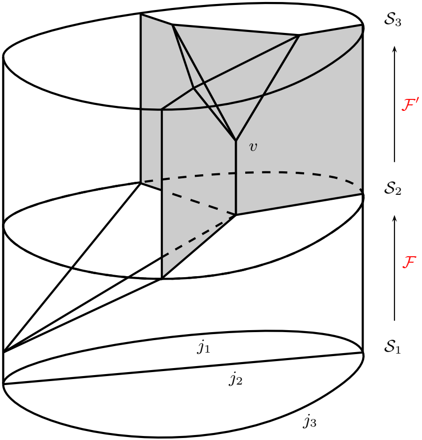


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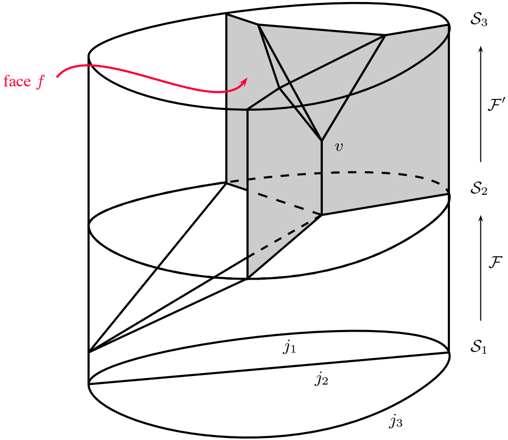


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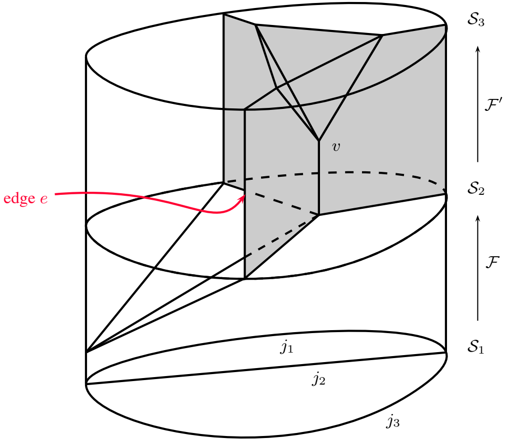


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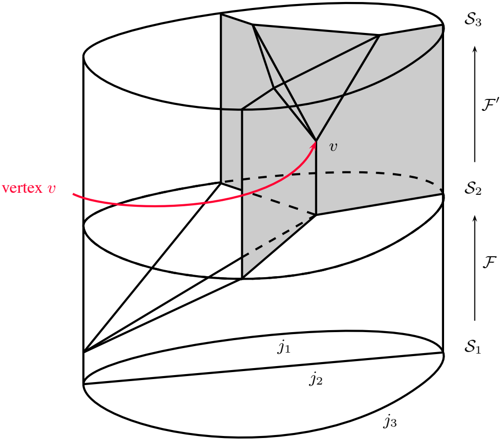


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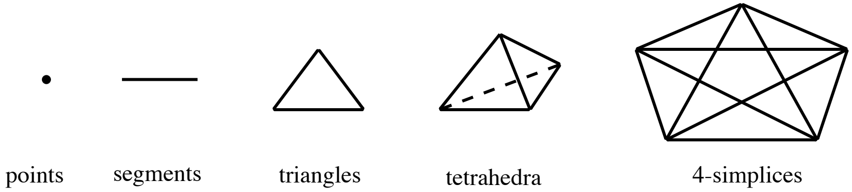


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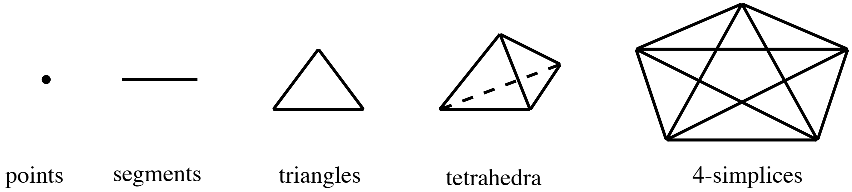


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What is the relation with the Hamiltonian constraint operator $\widehat{H}$?

We want to compute the physical inner product

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where the projector

$$\mathcal{P}_H = \int \mathcal{D}[N] \exp\left( i \int_\Sigma d^3x \, N(x) \widehat{H}(x) \right)$$

projects onto the kernel of the quantum Hamiltonian constraint.

What is the relation with the Hamiltonian constraint operator $\widehat{H}$?

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With an appropriate regularization $\mathcal{P}_H^\alpha$ of the projector, it is possible to formally write the inner product as

$$\langle \mathcal{S}_1 | \mathcal{P}_H^\alpha | \mathcal{S}_2 \rangle_{\mathrm{kin}} = \sum_{n=0}^{\infty} \frac{i^n \alpha^n}{n!} \sum_{\mathcal{F}_n : \mathcal{S}_1 \to \mathcal{S}_2} \prod_v A_v,$$

where $\mathcal{F}_n$ is a spin foam with $n$ vertices, i.e. $n$ actions of the Hamiltonian constraint.

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Gravity in any dimension $D$ can be written as a constrained topological field theory for the structure group $\mathrm{SO}(D)$ or $\mathrm{SO}(D-1,1)$.

This is encoded in the Plebanski action

$$S[\omega, B, \Phi] = \int_{\mathcal{M}} \left( B^{IJ} \wedge F_{IJ} + \frac{1}{\gamma} \star B^{IJ} \wedge F_{IJ} - \frac{1}{2} \Phi_{IJKL} B^{IJ} \wedge B^{KL} \right),$$

where $B$ is a Lie algebra valued 2-form, $F$ a curvature 2-form, and $\Phi$ a multiplier used to enforce the simplicity constraint.

This simplicity constraint ensures that the $B$ field can be written as

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The idea is that the simplicity constraint is going to give restions on the group representations that enter the definition of the amplitude.

Several models on the market:

- BC (Barrett, Crane), 4D, $\Lambda = 0$
- FK (Freidel, Krasnov), 4D, $\Lambda = 0$
- EPRL (Engel, Pereira, Rovelli, Livine), 4D, $\Lambda = 0$
- Crane-Yetter, 4D, $\Lambda \neq 0$
- Ponzano-Regge, 3D, $\Lambda = 0$
- Turaev-Viro, 3D, $\Lambda \neq 0$
- ...

How to extract predictions from the covariant theory?
With the boundary formalism (Oeckl, Conrady, Rovelli,. . . ).

Start with the boundary functional

$$W[\varphi, \Sigma] = \int_{\phi|_\Sigma = \varphi} \mathcal{D}[\phi] \exp\left(iS[\phi]\right).$$

$W[\varphi, \Sigma] = W[\varphi]$ because of diffeomorphism invariance. Then we can compute

$$W[x, y, \Psi] = \int \mathcal{D}[\varphi]\, \varphi(x)\varphi(y) W[\varphi] \Psi[\varphi].$$

The information about the geometry of the surface $\Sigma$ is in the gravitational field $\varphi$. We need to specify a state $\Psi[\varphi]$ peaked on some semi-classical geometry on the boundary.

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What can we learn from cosmology?

1920' In an homogeneous and isotropic model (FLRW), volume goes to zero and curvature to infinity as we go back in time: big bang singularity!

1950' Singularities are an artifact due to symmetry reduction (Gamov, Khalatnikov, Lifshitz,. . . )

1960' Singularities are ubiquitous (Hawking, Penrose).

Are singularities resolved in quantum gravity?

Loop quantum cosmology: apply the quantizations techniques of loop quantum gravity to cosmological models. With $c \propto \dot{a}$ and $p \propto a^2$, the Hamiltonian is

$$H = -\frac{3}{\gamma^2}\sqrt{p}c^2 + \frac{1}{2}\frac{p_\phi}{p^{3/2}} = 0.$$

To extract physics, it is usefull (although not necessary!) to deparametrize the system and use the matter field as a relational time. We can write down observables such as the volume $v(\phi)$ of the Universe.

The naive Wheeler-DeWitt quantization (i.e. with Schrödinger representation)

$$\hat{c}\Psi = c\Psi, \qquad \hat{p}\Psi = -i\hbar\frac{\partial}{\partial c}\Psi,$$

without additional input, leads to a quantum theory in which the singularity is not resolved.

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Stone–Von Neumann's uniqueness theorem.

Imagine a particle living in $\mathbb{R}$ with coordinates on phase space $T^*\mathbb{R} = \mathbb{R}^2$ given by $q$ and $p$. The Weyl–Heisenberg algebra is obtained by taking finite linear combinations of the generators

$$Q(\alpha) = e^{i\alpha q}, \qquad P(\beta) = e^{i\beta p}.$$

Uniqueness theorem: every representation of this algebra that is weakly continuous in the parameters $\alpha$ and $\beta$ is unitarily equivalent to the Schrödinger representation on $L^2(\mathbb{R}, dq)$:

$$\widehat{Q}(\alpha)\Psi(q) = e^{i\alpha q}\Psi(q), \qquad \widehat{P}(\beta)\Psi(q) = \Psi(q+\beta).$$

Weak continuity ensures that there exist self-adjoint operators $\hat{q}$ and $\hat{p}$ such that

$$\hat{q}\Psi(q) = q\Psi(q), \qquad \hat{p}\Psi(q) = -i\frac{\partial}{\partial q}\Psi(q).$$

(Can be obtained through GNS with a particular state $F$.)

How can we obtain different inequivalent representations? By relaxing some hypothesis of the theorem, such as the continuity. If $P(\beta)$ is not continuous in $\beta$, there is no self-adjoint operator

$$\hat{p} = -i\hbar \frac{\partial}{\partial q}.$$

This is indeed what we can expect if the spatial continuum picture fails!

Can we still do quantum mechanics? Yes, on the Hilbert space

$$\mathcal{H}_{\text{poly}} = L^2(\mathbb{R}_d, \mu_d).$$

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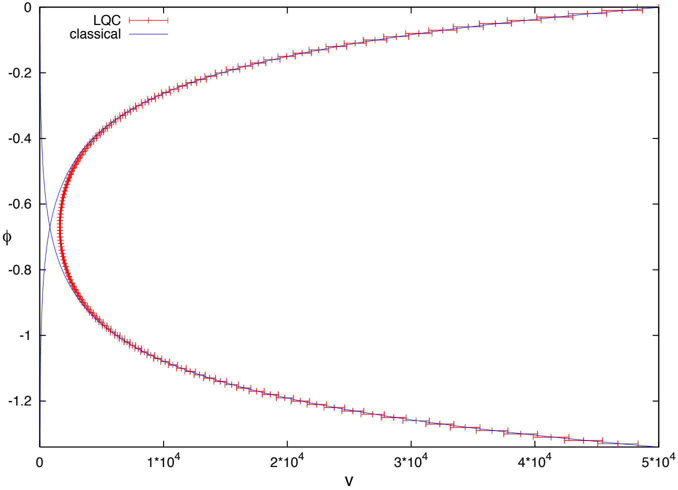
In FLRW loop quantum cosmology, we evaluate the holonomy of the connection along straight edges $\mu$ and obtain

$$h_\mu(c) = \cos\left(\frac{\mu c}{2}\right)\mathbb{I} + 2\sin\left(\frac{\mu c}{2}\right)\tau_i.$$

We end up working with the algebra of almost-periodic functions

$$\mathrm{Cyl} \ni g(c) = \sum_j \alpha_j \exp\left(i\frac{\mu_j c}{2}\right)$$




The Universe does not reached the singularity but is bounced back.

It is possible to derive the leading order quantum correction to the Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right)$$

where $\rho_c \approx 0.82\rho_{\mathrm{Pl}}$.

Some science-fiction: what happened before our Universe?

- Another classical Universe?
- A quantum mess?
- Nothing?

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# Conclusion

A provocative overstatement:
Loop quantum gravity is apparently able to provide “some” notion of what quantum general relativity might look like. The key idea is that quantum general relativity is in fact quantum geometry, a theory of quantum space-time.

A key role is played by background independence.